Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
European Journal of Mechanics B/Fluids 22 (2003) 39–50
EUROPEAN JOURNAL OF MECHANICS B/FLUIDS

# Instability of a nearly inextensible thin layer in a shear flow

François Charru [a,*], Paolo Luchini [b], Patricia Ern [a]

[a] *Institut de mécanique des fluides de Toulouse, allée du Professeur Camille Soula, 31400 Toulouse, France*
[b] *Dipartimento di Ingegneria Meccanica, Università di Salerno, 84084 Fisciano (SA), Italy*

Received 20 October 2002; accepted 9 January 2003

**Abstract**

The stability of a nearly inextensible thin layer in a shear flow is investigated using an adjoint-based approach of the perturbed Stokes flow, i.e., for small inertia effects. The thin layer may be a very viscous fluid or an elastic sheet. We show that long waves are unstable, and that short waves are also unstable when the effect of inertia dominates that of surface tension. In contrast with the two-layer flow with a clean interface, the instability persists even when the fluids on either side of the thin layer have the same viscosity.


*Keywords:* Hydrodynamic stability; Interfacial waves; Viscous flow

## 1. Introduction

We consider the stability of the viscous shear flow of two superposed layers (Fig. 1), between which a thin third layer is placed. The in-between layer may be a very viscous fluid, an elastic sheet or any nearly inextensible thin layer, as discussed below.

This three-layer problem is related to that of the two-layer shear flow with a "clean" interface. The two-layer flow is known to be unstable owing to the viscosity jump at the interface, however small the Reynolds number. Short waves are unstable [1], as well as long waves [2], especially when the thinner layer is the more viscous. The mechanism of the instability is based on small inertia effects which create a small disturbance flow out of phase with the interface disturbance [3,4].

The aim of this paper is twofold. First, we want to discuss how the interfacial inertial instability of the classical two-layer flow is modified by the thin third layer. Second, we want to show that the stability problem can be solved by an adjoint-based approach, which may need less calculations than the classical direct method.

The paper is organized as follows. In Section 2, the model is presented and the physical situations to which it may correspond are discussed. In Section 3, the adjoint-based approach is presented. Sections 4 and 5 are devoted to obtaining the wave velocity and growth rate of long and short waves, respectively. The results are discussed in the final section.

## 2. The model

We want to determine the stability of the three-layer flow in the particular case where the in-between layer is thin enough, so that the three-layer problem can be simplified to an equivalent two-layer problem. Then, differences with the classical two-layer problem (i.e., without any in-between thin layer) only arise in the boundary conditions at the deformable interface. The precise conditions for the two-layer simplification are first derived, and then the set of stability equations to be solved is given.

* Corresponding author.
*E-mail addresses:* charru@imft.fr (F. Charru), luchini@unisa.it (P. Luchini).

0997-7546/03/$ – see front matter 
doi:10.1016/S0997-7546(03)00002-5

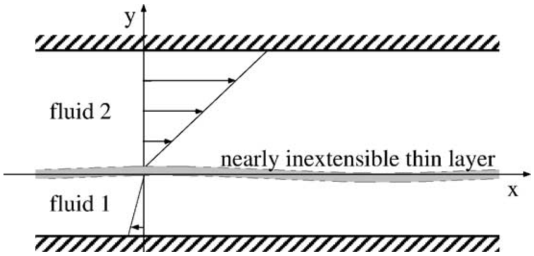


Fig. 1. Sketch of the thin layer in a shear flow.

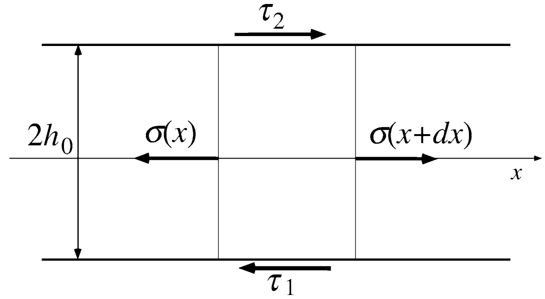


Fig. 2. Longitudinal stress balance in the thin layer, with $\sigma = -p_0 + 2ik\mu_0 u_0$ for a viscous layer and $\sigma = -E_0 u_0/c$ for an elastic sheet.

The lower and upper fluid layers have viscosity $\mu_1$ and $\mu_2$, and thickness $h_1$ and $h_2$. In the reference frame moving with the undisturbed thin layer, the base flow is taken to be given by:

$$U_1 = \gamma_1 y, \qquad U_2 = \gamma_2 y, \tag{1}$$

(Fig. 1) where the shear rates $\gamma_{1,2}$ satisfy shear stress continuity, $\mu_1\gamma_1 = \mu_2\gamma_2$. This base flow is based on the assumption that the thickness $2h_0$ of the in-between layer is much smaller than $h_{1,2}$, so that $y \pm h_0 \approx y$, and ignores any motion in the thin layer. The latter assumption is exact for an elastic sheet, and for a viscous fluid layer with viscosity $\mu_0$ it amounts to neglecting velocities of order $U_0 \sim h_0\gamma_{1,2}\mu_{1,2}/\mu_0$.

### 2.1. Stress balance in the thin layer

For the classical two-layer problem, the conditions at the interface impose continuity of the velocity and of the tangential and normal stresses (see [2] for example). For an in-between layer with thickness much smaller than any other length, velocity is still continuous across the layer, as well as normal stresses. In order for a tangential-stress condition to be established, the longitudinal equilibrium of the thin layer (Fig. 2) must be analysed. Let us consider two-dimensional disturbances of the base flow (1), proportional to $e^{ik(x-ct)}$ where $k$ is the wavenumber and $c$ the complex wave velocity. Consider first the case of a very viscous thin layer. Neglecting inertia and integrating the local tangential equilibrium condition over the thickness $2h_0$ gives for the complex amplitudes of the disturbances:

$$ik(-p_0 + 2ik\mu_0 u_0) = \frac{\tau_2 - \tau_1}{2h_0}, \tag{2}$$

where $p_0$ and $u_0$ are the pressure and the longitudinal velocity in the thin layer, and $\tau_{1,2}$ are the tangential stresses at the upper and lower interfaces. The pressure term can be neglected in the above stress balance. Indeed, it will be shown in Section 4.3 that, for long waves, the pressure and shear stress are tied to the interface displacement $\eta$ as $p_0 \sim \mu_{1,2}\gamma_{1,2}\eta/kh_{1,2}^2$ and $\tau_{1,2} \sim \mu_{1,2}\gamma_{1,2}\eta/h_{1,2}$, so that the pressure term can be neglected as soon as the thin layer satisfies $h_0/h_{1,2} \ll kh_{1,2}$. For short waves, it will be shown in Section 5.3 that $p_0 \sim \tau_{1,2} \sim \mu_{1,2}k\gamma_{1,2}\eta$, so that neglecting the pressure term requires $kh_0 \ll 1$.

Thus, (2) simplifies to:

$$\mu_0 h_0 k^2 u_0 \sim \tau_2 - \tau_1. \tag{3}$$

For $k = 0$ or an interface with zero thickness, the latter equation implies shear stress continuity, i.e., the classical boundary condition for a clean interface. In the opposite limit, i.e., nonzero $kh_0$ and a very viscous thin layer, since the tangential stress jump must remain finite, the above condition implies at the dominant order that the velocity $u_0$ is zero, so that the tangential stress balance reduces to:

$$u_0 = U_1(-h_0 + \eta) = U_2(h_0 + \eta) = 0. \tag{4}$$

More precisely, with the above estimates of the shear stress disturbances, the zero velocity condition (4) holds when $u_0 \ll u_{1,2}$, i.e., when $(kh_{1,2})(kh_0)\mu_0/\mu_{1,2} \gg 1$ for long waves, and when $(kh_0)\mu_0/\mu_{1,2} \gg 1$ for short waves.

Consider now the case of a thin elastic sheet with Young modulus $E_0$. The stress balance now gives:

$$E_0 h_0 k \frac{u_0}{c} \sim \tau_2 - \tau_1, \tag{5}$$

where $u_0/(-ikc)$ is the displacement and $-u_0/c$ is the deformation. As for a viscous thin layer, the two extreme cases are (I) no thin sheet, which implies shear stress continuity, and (II) a nearly inextensible sheet, which implies $u_0 = 0$, i.e., the zero velocity condition (4). More precisely, this condition holds as long as $E_0 k h_0/(\mu_{1,2} h_{1,2} c) \gg 1$, where the wave velocity will be shown below to be $c \sim \gamma_{1,2} h_{1,2}$ for long waves, and $c \sim \rho_{1,2}\gamma_{1,2}^2/(\mu_{1,2} k^3)$ for short waves.

From now on, consistently with the assumption that the thin layer thickness be much smaller than any other length, the interfacial conditions will be written at $y = \eta$ rather than at $y = \pm h_0 + \eta$, and linearized about $y = 0$.

### 2.2. Stability equations

On considering two-dimensional disturbances of the base flow (1), the stability equations are obtained by linearising the disturbance equations around the base flow, and searching for normal modes proportional to $\mathrm{e}^{\mathrm{i}k(x-ct)}$. The streamfunction being defined by $u_{1,2} = \psi'_{1,2}(y)$ and $v_{1,2} = -\mathrm{i}k\psi_{1,2}(y)$, the vorticity equation becomes the Orr–Sommerfeld equation:

$$\mu_{1,2}\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - k^2\right)^2 \psi_{1,2} = \mathrm{i}k\rho_{1,2}\big[(U_{1,2} - c)\big(\psi''_{1,2} - k^2\psi_{1,2}\big) - U''_{1,2}\psi_{1,2}\big]. \tag{6}$$

At the upper and lower walls, the no-slip conditions are:

$$\psi_1(-h_1) = \psi'_1(-h_1) = 0, \tag{7a}$$

$$\psi_2(h_2) = \psi'_2(h_2) = 0. \tag{7b}$$

For a thin, nearly inextensible, in-between layer, continuity of the velocity still holds as well as continuity of normal stresses, and the longitudinal equilibrium of the thin sheet reduces, as discussed above, to the zero-velocity condition (4). All together the interfacial conditions are:

$$\psi_1(0) = \psi_2(0), \tag{8a}$$

$$U'_1\eta + \psi'_1(0) = U'_2\eta + \psi'_2(0), \tag{8b}$$

$$U'_1\eta + \psi'_1(0) = 0, \tag{8c}$$

$$p_1(0) + 2\mathrm{i}k\mu_1\psi'_1(0) - p_2(0) - 2\mathrm{i}k\mu_2\psi'_2(0) = Tk^2\eta, \tag{8d}$$

together with the kinematic (or no mass transfer) condition:

$$\psi_1(0) = c\eta. \tag{9}$$

The pressure disturbances $p_{1,2}$ are related to the streamfunctions by the momentum equations:

$$\mathrm{i}kp_{1,2} = \mu_{1,2}\big(\psi'''_{1,2} - k^2\psi'_{1,2}\big) + \mathrm{i}k\rho_{1,2}\big[(c - U_{1,2})\psi'_{1,2} + U'_{1,2}\psi_{1,2}\big], \tag{10}$$

with $U_{1,2}(0) = 0$. For a very viscous sheet, $T$ is the sum of the surface tensions at the two interfaces; for an elastic sheet, $T$ represents the tension of the sheet per unit length. Note that (I) the tangential-stress jump induces small variations of the tension along the sheet, but the corresponding correction in the normal stress condition is of the second order with respect to perturbation amplitude, and (II) the increase of the elastic tension due to the increase of the length of the deformed sheet is also of second order. Finally note that, as a consequence of (8b), (8c) and (9), the inertia terms in (10) on either side exactly cancel each other at the interface.

## 3. Adjoint-based perturbation expansion

The adjoint-based approach to finding the eigenvalues of a perturbed operator or matrix is a well-known fundamental tool of quantum-mechanical perturbation theory (where in addition, all the relevant operators being hermitian and self-adjoint, left-hand or adjoint eigenfunctions coincide with complex conjugates of the right-hand or direct eigenfunctions). A similar approach can be advantageously applied to hydrodynamic stability problems (see, e.g., [5, p. 15]). Let us consider a general (not necessarily self-adjoint) matrix $\mathbf{A}$ depending on an eigenvalue parameter $\lambda$; this includes as particular cases the standard eigenvalue problem where $\mathbf{A}(\lambda) = \mathbf{A}(0) - \lambda\mathbf{I}$ and the generalized eigenvalue problem where $\mathbf{A}(\lambda) = \mathbf{A}(0) - \lambda\mathbf{B}$, but a more general nonlinear dependence on $\lambda$ may also be assumed. Its eigenvalues $\lambda$ and eigenvectors $\mathbf{u}$ will be solutions of

$$\mathbf{A}(\lambda)\mathbf{u} = 0.$$

Let us now add the assumption that $\mathbf{A}$ also depends on a small perturbation parameter $\varepsilon$, and let us look for the eigenvalues and eigenvectors of $\mathbf{A}(\lambda, \varepsilon)$ in the form of a regular perturbation expansion where $\mathbf{u} = \mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)} + \cdots$ and $\lambda = \lambda^{(0)} + \varepsilon\lambda^{(1)} + \cdots$.

Inserting such expansions into the eigenvector equation and collecting equal powers of $\varepsilon$ yields the following hierarchy of equations:

$$\mathbf{A}\big(\lambda^{(0)},0\big)\mathbf{u}^{(0)} = 0, \tag{11a}$$

$$\mathbf{A}\big(\lambda^{(0)},0\big)\mathbf{u}^{(1)} = -\lambda^{(1)}\frac{\partial\mathbf{A}}{\partial\lambda}\mathbf{u}^{(0)} - \frac{\partial\mathbf{A}}{\partial\varepsilon}\mathbf{u}^{(0)}, \tag{11b}$$

$$\mathbf{A}\big(\lambda^{(0)},0\big)\mathbf{u}^{(2)} = -\lambda^{(2)}\frac{\partial\mathbf{A}}{\partial\lambda}\mathbf{u}^{(0)} + \cdots, \tag{11c}$$

$$\cdots$$

A straightforward perturbation approach involves inverting this hierarchy of equations in a sequence, where account must be taken of the fact that the operator on the l.h.s. (always the same at every order) is singular and a compatibility condition is involved. However, whereas to do so is relatively easy if the operators involved are matrices, the solution becomes more and more complicated at every step if differential operators are involved and an analytical solution is sought. On the other hand, if the eigenvalue perturbation only is the goal, the simpler possibility exists of imposing the compatibility condition alone, without actually determining the first-order correction to the eigenfunction. If the unperturbed eigenvalue is non-degenerate, this compatibility condition is simply given by

$$\left(\mathbf{v}^{(0)}\frac{\partial\mathbf{A}}{\partial\lambda}\mathbf{u}^{(0)}\right)\lambda^{(1)} = -\mathbf{v}^{(0)}\frac{\partial\mathbf{A}}{\partial\varepsilon}\mathbf{u}^{(0)}, \tag{12}$$

where $\mathbf{v}^{(0)}$ is the left-hand eigenfunction of the unperturbed $\mathbf{A}(\lambda^{(0)},0)$ operator, i.e., the particular vector or function such that $\mathbf{v}^{(0)}\,\mathbf{A}(\lambda^{(0)},0) = 0$. That Eq. (12) is necessary is easily proved by left-multiplying both sides of Eq. (11b) with $\mathbf{v}^{(0)}$. When the eigenvalue is non-degenerate, i.e., the rank of matrix $\mathbf{A}(\lambda^{(0)},0)$ is deficient by exactly 1, this condition is also sufficient.

## 4. Long-wave instability

### *4.1. The direct and adjoint problems*

The long-wave stability problem corresponds to disturbances with wavelength much longer than each layer's thickness, i.e., $\alpha = kh_1 \ll 1$, and to small inertia effects, i.e., $\alpha Re_{\mathrm{lw}} \ll 1$ (we shall assume $Re_{\mathrm{lw}} \sim 1$), where the Reynolds number is defined by:

$$Re_{\mathrm{lw}} = \frac{\rho_1\gamma_1 h_1^2}{\mu_1}. \tag{13}$$

Once $\gamma_1^{-1}$ and $h_1$ are chosen as the time and length scales, the problem also depends on the density, viscosity and thickness ratios and on the Weber number, defined as:

$$r = \frac{\rho_2}{\rho_1}, \qquad m = \frac{\mu_2}{\mu_1}, \qquad d = \frac{h_2}{h_1}, \qquad We_{\mathrm{lw}} = \frac{\rho_1\gamma_1^2 h_1}{Tk^2}. \tag{14}$$

We shall assume that $We_{\mathrm{lw}} \sim 1$, so that the stabilising effect of surface tension arises at the same order as inertia. If in Eq. (6) with boundary conditions (7)–(9) both the unknowns and the eigenvalue are expanded in a series of powers of the small dimensionless wavenumber $\alpha$, the following hierarchy is obtained:

Order 0

$$\psi_{1,2}^{(0)''''} = 0, \tag{15a}$$

$$\psi_1^{(0)}(-1) = \psi_1^{(0)'}(-1) = 0, \tag{15b}$$

$$\psi_2^{(0)}(d) = \psi_2^{(0)'}(d) = 0, \tag{15c}$$

$$\psi_1^{(0)}(0) - \psi_2^{(0)}(0) = 0, \tag{15d}$$

$$\psi_1^{(0)}(0) - c^{(0)}\eta^{(0)} = 0, \tag{15e}$$

$$\eta^{(0)} + \psi_1^{(0)'}(0) - m^{-1}\eta^{(0)} - \psi_2^{(0)'}(0) = 0, \tag{15f}$$

$$\eta^{(0)} + \psi_1^{(0)'}(0) = 0, \tag{15g}$$

$$\psi_1^{(0)'''}(0) - m\psi_2^{(0)'''}(0) = 0. \tag{15h}$$

Order 1

$$\psi_1^{(1)''''} = \mathrm{i}Re_{\mathrm{lw}}\big(y - c^{(0)}\big)\psi_1^{(0)''}, \tag{16a}$$

$$\psi_2^{(1)''''} = \frac{\mathrm{i}rRe_{\mathrm{lw}}}{m}\left(\frac{y}{m} - c^{(0)}\right)\psi_2^{(0)''}, \tag{16b}$$

$$\psi_1^{(1)}(-1) = \psi_1^{(1)'}(-1) = 0, \tag{16c}$$

$$\psi_2^{(1)}(d) = \psi_2^{(1)'}(d) = 0, \tag{16d}$$

$$\psi_1^{(1)}(0) - \psi_2^{(1)}(0) = 0, \tag{16e}$$

$$\psi_1^{(1)}(0) - c^{(0)}\eta^{(1)} = c^{(1)}\eta^{(0)}, \tag{16f}$$

$$\eta^{(1)} + \psi_1^{(1)'}(0) - m^{-1}\eta^{(1)} - \psi_2^{(1)'}(0) = 0, \tag{16g}$$

$$\eta^{(1)} + \psi_1^{(1)'}(0) = 0, \tag{16h}$$

$$\psi_1^{(1)'''}(0) - m\psi_2^{(1)'''}(0) = \mathrm{i}\eta^{(0)}\frac{Re_{\mathrm{lw}}}{We_{\mathrm{lw}}}. \tag{16i}$$

In order to obtain the eigenvalue perturbation without actually solving the order-1 equations, the method of Section 3 can now be applied. In so doing, care must be taken of the fact that the unknowns of the problem include both the continuous function $\psi_{1,2}$ and the discrete variable $\eta$. The adjoint system of the order-0 equations (15a) can be obtained by constructing a weighted sum of the equations and boundary conditions. Actually, each boundary condition can either be considered as part of the definition of the functions $\psi_{1,2}$, and thus implicitly imposed, or as an explicit equation with an associated adjoint variable. The two choices are equivalent, and do not in any way alter the result. In practice, it is convenient to impose simple boundary conditions that remain unchanged at all orders implicitly, by requiring that $\psi_{1,2}^{(n)}$ and $\eta^{(n)}$ obey them for all $n$, and to introduce adjoint variables only for those boundary conditions that exhibit a non-zero r.h.s. in the order-1 equations. Accordingly we construct the weighted sum

$$\int_{-1}^{d} \varphi_{1,2}(y)\psi_{1,2}^{(0)''''}(y)\,\mathrm{d}y + \zeta\big(\psi_1^{(0)}(0) - c^{(0)}\eta^{(0)}\big) + \xi\big(\psi_1^{(0)'''}(0) - m\psi_2^{(0)'''}(0)\big) \equiv 0, \tag{17}$$

and impose that this identity be satisfied for arbitrary $\eta^{(0)}$ and $\psi^{(0)}(y)$ obeying all the other boundary conditions but not (15e) and (15h) nor the differential equation (15a). An integration by parts then gives

$$\begin{aligned}
&\int_{-1}^{d} \varphi_{1,2}(y)''''\psi_{1,2}^{(0)}(y)\,\mathrm{d}y + \varphi_2(d)\psi_2^{(0)'''}(d) - \varphi_2'(d)\psi_2^{(0)''}(d) + \varphi_2''(d)\psi_2^{(0)'}(d) - \varphi_2'''(d)\psi_2^{(0)}(d) - \varphi_2(0)\psi_2^{(0)'''}(0)\\
&\quad + \varphi_2'(0)\psi_2^{(0)''}(0) - \varphi_2''(0)\psi_2^{(0)'}(0) + \varphi_2'''(0)\psi_2^{(0)}(0) + \varphi_1(0)\psi_1^{(0)'''}(0) - \varphi_1'(0)\psi_1^{(0)''}(0) + \varphi_1''(0)\psi_1^{(0)'}(0)\\
&\quad - \varphi_1'''(0)\psi_1^{(0)}(0) - \varphi_1(-1)\psi_1^{(0)'''}(-1) + \varphi_1'(-1)\psi_1^{(0)''}(-1) - \varphi_1''(-1)\psi_1^{(0)'}(-1) + \varphi_1'''(-1)\psi_1^{(0)}(-1)\\
&\quad + \zeta\big(\psi_1^{(0)}(0) - c^{(0)}\eta^{(0)}\big) + \xi\big(\psi_1^{(0)'''}(0) - m\psi_2^{(0)'''}(0)\big) \equiv 0,
\end{aligned} \tag{18}$$

whence, on eliminating $\psi_2^{(0)}(d),\ \psi_2^{(0)'}(d),\ \psi_1^{(0)}(-1),\ \psi_1^{(0)'}(-1),\ \psi_2^{(0)}(0),\ \psi_1^{(0)'}(0)$, and $\psi_2^{(0)'}(0)$ through the boundary conditions, and separately equating to zero the coefficients of $\psi_2^{(0)''}(d),\ \psi_2^{(0)'''}(d),\ \psi_1^{(0)''}(-1),\ \psi_1^{(0)'''}(-1),\ \psi_1^{(0)}(0),\ \psi_1^{(0)''}(0),$ $\psi_2^{(0)''}(0),\ \psi_1^{(0)'''}(0),\ \psi_2^{(0)'''}(0)$ and $\eta^{(0)}$, the following adjoint system of equations and boundary conditions is obtained:

$$\varphi_{1,2}'''' = 0, \tag{19a}$$

$$\varphi_1(-1) = \varphi_1'(-1) = 0, \tag{19b}$$

$$\varphi_2(d) = \varphi_2'(d) = 0, \tag{19c}$$

$$\varphi_2(0) + m\xi = 0, \tag{19d}$$

$$\varphi_1(0) + \xi = 0, \tag{19e}$$

$$\varphi_1'(0) = \varphi_2'(0) = 0, \tag{19f}$$

$$m^{-1}\varphi_2''(0) - \varphi_1''(0) - c^{(0)}\zeta = 0, \tag{19g}$$

$$\varphi_2'''(0) - \varphi_1'''(0) + \zeta = 0. \tag{19h}$$

*4.2. Wave velocity and growth rate*

The solution of the direct problem at order zero gives the Stokes flow:

$$\psi_1^{(0)} = -(y+1)^2\big(\big(2c^{(0)}+1\big)y - c^{(0)}\big)\eta^{(0)}, \tag{20a}$$

$$\psi_2^{(0)} = -(y-d)^2\left(\frac{(2c^{(0)}+1)y}{m} - \frac{c^{(0)}}{d^2}\right)\eta^{(0)}, \tag{20b}$$

and the solution of the adjoint problem is:

$$\varphi_1 = -(y+1)^2(1-2y)\xi, \tag{21a}$$

$$\varphi_2 = -\frac{3m}{d^2}(y-d)^2\left(\frac{2y}{3d} + \frac{1}{3}\right)\xi, \tag{21b}$$

$$\zeta = 12\frac{m+d^3}{d^3}\xi. \tag{21c}$$

Both homogeneous eigenvalue problems, as they should, give the same wave velocity $c^{(0)}$:

$$c^{(0)} = -\frac{1}{2}\frac{d(d^2-1)}{m+d^3}. \tag{22}$$

The wave velocity is positive when $d < 1$ and negative otherwise: the wave propagates with the flow of the thinner layer. This result is in contrast with the clean interface case, where the wave propagates in the direction of the less viscous flow, no matter its thickness.

Once the direct and adjoint problems have been solved, use can be made of the properties of (17). The adjoint has been constructed so that (17) not only applies to $\psi^{(0)}$ and $\eta^{(0)}$, but to any like couple obeying all the other boundary conditions but not (15e) and (15h) nor the differential equation (15a). One such couple is $\psi^{(1)}$, $\eta^{(1)}$. Eq. (17), when applied to the order-1 equations (16a) and (16b), gives the compatibility condition:

$$\int_{-1}^{0}\varphi_1(y)\mathrm{i}Re_{\mathrm{lw}}\big(y-c^{(0)}\big)\psi_1^{(0)\prime\prime}\,\mathrm{d}y + \int_{0}^{d}\varphi_2(y)\frac{\mathrm{i}rRe_{\mathrm{lw}}}{m}\left(\frac{y}{m} - c^{(0)}\right)\psi_2^{(0)\prime\prime}\,\mathrm{d}y + \zeta c^{(1)}\eta^{(0)} + \xi\mathrm{i}\alpha^2\eta^{(0)}\frac{Re_{\mathrm{lw}}}{We_{\mathrm{lw}}} = 0.$$

Thence the order-1 correction of the eigenvalue, $c^{(1)}$, due to both inertial and capillary effects, is obtained as:

$$\begin{aligned} c^{(1)} = {} & \frac{\mathrm{i}}{120}\frac{Re_{\mathrm{lw}}\,d^3}{m^2(m+d^3)^3}\big(6m^3d - m^2d^6 - 2m^3d^3 + 3m^2d^2 + 2m^4 + r\big(2d^7 + 6d^6m - 2d^4m + 3d^5m^2 - dm^2\big)\big) \\ & - \frac{\mathrm{i}(kh_1)^2d^3}{4(m+d^3)}\frac{Re_{\mathrm{lw}}}{We_{\mathrm{lw}}}. \end{aligned} \tag{23}$$

The dimensional growth rate is given by $\sigma = -\mathrm{i}\gamma_1(kh_1)^2c^{(1)}$. We have verified that the above expression satisfies the following symmetry relation, which corresponds to the permutation of the subscripts 1 and 2 defining the lower and upper fluids:

$$c^{(1)}(m,d,r) = \frac{rd^4}{m^3}c^{(1)}\left(\frac{1}{m}, \frac{1}{d}, \frac{1}{r}\right). \tag{24}$$

Fig. 3 displays curves of iso-values of $c^{(1)}/\mathrm{i}Re_{\mathrm{lw}}$ in the $(m,d)$ plane for equal density ($r = 1$) and zero surface tension ($We_{\mathrm{lw}} = \infty$). It appears that the flow is unstable for a wide range of the parameters. In particular,

- when the viscosities are equal, the flow is always unstable, unlike the clean-interface case for which the growth rate is zero [2];
- when one layer is much thinner and much less viscous than the other, the flow is stable as in the clean interface case.

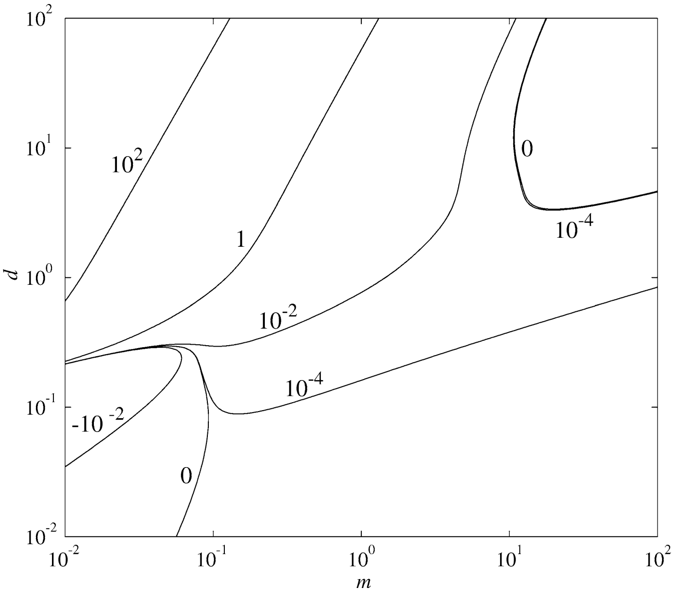


Fig. 3. Curves of isovalues of $c^{(1)}/\mathrm{i}Re_{\mathrm{lw}}$ in the plane ($m = \mu_2/\mu_1$, $d = h_2/h_1$), for equal density and zero surface tension (in the upper right corner, the curves $c^{(1)}/\mathrm{i}Re_{\mathrm{lw}} = 0$ and $10^{-4}$ are not distinguishable).

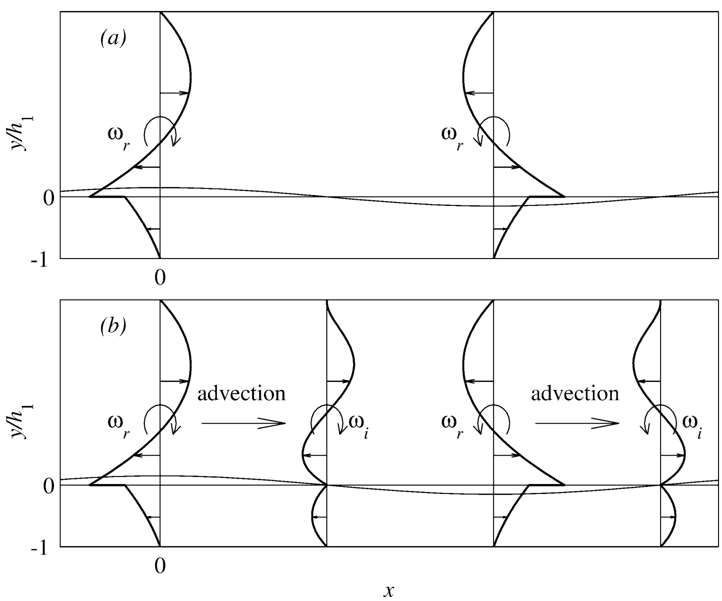


Fig. 4. Eigenfunctions $\psi'(y)$. (a) Dominant Stokes flow $\psi^{(0)\prime}(y)$; (b) Inertial correction $\psi^{(1)\prime}(y)$. Parameters: $m = 0.5$, $d = 3$, $r = 1$, $We_{\mathrm{lw}} = \infty$.

### 4.3. Mechanism of the long-wave instability

The above results show that for a wide range of viscosity ratios, including $m = 1$, the flow is unstable whatever the thickness ratio. This result is in contrast with the case of a clean interface, for which the flow is unstable if and only if the more viscous layer is thinner than the other. The aim of this section is to understand this difference, by comparing the mechanism of the instability of a nearly inextensible layer to that of a clean interface, which has been given in [4]. This mechanism can be described with the help of Fig. 4, which displays typical eigenfunctions for the longitudinal velocities $\psi'_{1,2}(y)$. (The inertial corrections of the eigenfunctions require the complete solution of the order-1 equations (16a)–(16i) of the direct problem, and are given in Appendix A. This calculation gives of course the same eigenvalue correction as the calculation using the adjoint problem, but at a greater expense of time!)

Consider the case when the thickness of one layer is much smaller than the other, say $h_1 \ll h_2$. The deformation of the interface creates in each fluid a velocity disturbance $u_{1,2}^{(0)} \sim \gamma_{1,2}\eta^{(0)}$ as shown in Fig. 4(a), to which there corresponds vorticity disturbances $\omega_{1,2}^{(0)} \sim -u_{1,2}^{(0)}/h_{1,2}$. This dominant Stokes flow is in phase with the deformation of the interface.

In the thick upper layer, advection by the base flow of the vorticity disturbance creates a small out-of-phase vorticity component $\omega_2^{(1)} \sim Re_{2\mathrm{eff}}\omega_2^{(0)}$, where $Re_{2\mathrm{eff}} = kh_2\rho\gamma_2 h_2^2/\mu_2$ is the effective Reynolds number. The velocity field $u_2^{(1)} \sim \omega_2^{(1)} h_2$ associated with this out-of-phase vorticity must satisfy the no-slip condition at the interface, and therefore is as shown in Fig. 4(b). The out-of-phase velocity $u_2^{(1)}$ creates an interfacial shear stress $\tau_2^{(1)} \sim \mu_2 u_2^{(1)}/h_2$ and a pressure disturbance $p_{1,2}^{(1)} \sim \mu_2 u_2^{(1)}/kh_2^2$. In the lower thin layer, the inertial flow is smaller by a factor $(h_1/h_2)^3$ and is negligible. In this layer, the out-of-phase flow $u_1^{(1)}$ is driven by the pressure disturbance $p_{1,2}^{(1)}$, so that $u_1^{(1)} \sim kp_{1,2}^{(1)} h_1^2/\mu_1$. The divergence of this flow leads to the growth rate:

$$\sigma = \frac{v_1^{(1)}}{\eta^{(0)}} \sim kh_1 \frac{u_1^{(1)}}{\eta^{(0)}} \sim \gamma_1 \frac{kh_1 Re_{2\mathrm{eff}}}{d}. \tag{25}$$

The above estimate for the growth rate agrees with (23) for high $d$. Thus, it appears that the mechanism given above differs from that of a clean interface in two ways:

- The inertial flow in the thin layer is created by the pressure disturbance $p_{1,2}^{(1)}$, whereas it is created by the shear stress disturbance $\tau_{1,2}^{(1)}$ for the clean-interface case [4].
- The direction of the disturbance flow in either fluid does not change when the viscosity ratio crosses unity, whereas it reverses for the clean-interface problem.

## 5. Short-wave instability

### 5.1. *The direct and adjoint problems*

The short-wave stability problem corresponds to disturbances with wavelength much shorter than the layer thickness, and to small inertia effects. The latter condition implies wavelengths much shorter than the viscous length $l_{v1,2} = (\mu_{1,2}/\rho\gamma_{1,2}k)^{1/3}$, which represents the penetration length of vorticity disturbances created at the deformed interface [4]. These two conditions can be written as:

$$kh_{1,2} \gg 1, \qquad Re_{\mathrm{sw}} = \frac{\rho\gamma_1}{\mu_1 k^2} \ll 1, \tag{26}$$

where $Re_{\mathrm{sw}} = Re_{\mathrm{lw}}/\alpha^2$ is the appropriate Reynolds number for short waves. Once $\gamma_1^{-1}$ and $k^{-1}$ are chosen as the time and length scales, the problem also depends on the density and viscosity ratios and on the Weber number, defined as:

$$r = \frac{\rho_2}{\rho_1}, \qquad m = \frac{\mu_2}{\mu_1}, \qquad We_{\mathrm{sw}} = \frac{\rho_1\gamma_1^2}{Tk^3} = \frac{We_{\mathrm{lw}}}{\alpha}. \tag{27}$$

Expanding the streamfunctions, the interface position and the eigenvalue in a series of powers of the small Reynolds number $Re_{\mathrm{sw}}$, the following hierarchy is obtained for the Orr–Sommerfeld equation (6) and boundary conditions (7)–(9):

Order 0

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)^2 \psi_{1,2}^{(0)} = 0, \tag{28a}$$

$$\psi_1^{(0)}(-\infty) = \psi_1^{(0)\prime}(-\infty) = 0, \tag{28b}$$

$$\psi_2^{(0)}(\infty) = \psi_2^{(0)\prime}(\infty) = 0, \tag{28c}$$

$$\psi_1^{(0)}(0) - \psi_2^{(0)}(0) = 0, \tag{28d}$$

$$\psi_1^{(0)}(0) - c^{(0)}\eta^{(0)} = 0, \tag{28e}$$

$$\eta^{(0)} + \psi_1^{(0)\prime}(0) - \frac{\eta^{(0)}}{m} - \psi_2^{(0)\prime}(0) = 0, \tag{28f}$$

$$\eta^{(0)} + \psi_1^{(0)\prime}(0) = 0, \tag{28g}$$

$$\psi_1^{(0)\prime\prime\prime}(0) - m\psi_2^{(0)\prime\prime\prime}(0) = 0. \tag{28h}$$

Order 1

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)^2 \psi_1^{(1)} = \mathrm{i}\big(y - c^{(0)}\big)\big(\psi_1^{(0)\prime\prime} - \psi_1^{(0)}\big), \tag{29a}$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)^2 \psi_2^{(1)} = \frac{\mathrm{i}r}{m}\left(\frac{y}{m} - c^{(0)}\right)\big(\psi_2^{(0)\prime\prime} - \psi_2^{(0)}\big), \tag{29b}$$

$$\psi_1^{(1)}(-\infty) = \psi_1^{(1)\prime}(-\infty) = 0, \tag{29c}$$

$$\psi_2^{(1)}(\infty) = \psi_2^{(1)\prime}(\infty) = 0, \tag{29d}$$

$$\psi_1^{(1)}(0) - \psi_2^{(1)}(0) = 0, \tag{29e}$$

$$\psi_1^{(1)}(0) - c^{(0)}\eta^{(1)} = c^{(1)}\eta^{(0)}, \tag{29f}$$

$$\eta^{(1)} + \psi_1^{(1)\prime}(0) - \frac{\eta^{(1)}}{m} - \psi_2^{(1)\prime}(0) = 0, \tag{29g}$$

$$\eta^{(1)} + \psi_1^{(1)\prime}(0) = 0, \tag{29h}$$

$$\psi_1^{(1)\prime\prime\prime}(0) - m\psi_2^{(1)\prime\prime\prime}(0) = \frac{\mathrm{i}\eta^{(0)}}{We_{\mathrm{sw}}}. \tag{29i}$$

As for long waves, it was assumed that the stabilising effect of surface tension arises at the same order as inertia, i.e., that the effective Weber number $We_{\mathrm{sw}}$ is of order one.

The adjoint system of the order-0 equations (28a)–(28h) is obtained as for long waves by constructing the weighted sum

$$\int_{-\infty}^{\infty} \varphi_{1,2}(y)\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)^2 \psi_{1,2}^{(0)}(y)\,\mathrm{d}y + \zeta\left(\psi_1^{(0)}(0) - c^{(0)}\eta^{(0)}\right) + \xi\left(\psi_1^{(0)\prime\prime\prime}(0) - m\psi_2^{(0)\prime\prime\prime}(0)\right) \equiv 0, \tag{30}$$

integrating by parts, and imposing that this identity be satisfied for an arbitrary $\eta^{(0)}$ and $\psi^{(0)}(y)$ obeying the remaining boundary conditions only. The following adjoint system of equations and boundary conditions is obtained:

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)^2 \varphi_{1,2} = 0, \tag{31a}$$

$$\varphi_1(-\infty) = \varphi_1{}'(-\infty) = 0, \tag{31b}$$

$$\varphi_2(\infty) = \varphi_2{}'(\infty) = 0, \tag{31c}$$

$$\varphi_2(0) = m\varphi_1(0) = -m\xi, \tag{31d}$$

$$\varphi_2{}'(0) = \varphi_1{}'(0) = 0, \tag{31e}$$

$$\varphi_2{}''(0) - m\varphi_1{}''(0) = mc^{(0)}\zeta, \tag{31f}$$

$$\varphi_2{}'''(0) - \varphi_1{}'''(0) = -\zeta. \tag{31g}$$

Note that the interfacial boundary conditions are identical to those of the long wave problem.

### 5.2. *Wave velocity and growth rate*

The solution for the direct problem at order zero is found to be:

$$\psi_1^{(0)}(y) = -y\,\mathrm{e}^{y}\eta^{(0)}, \tag{32a}$$

$$\psi_2^{(0)}(y) = -\frac{1}{m}y\,\mathrm{e}^{-y}\eta^{(0)}, \tag{32b}$$

and for the adjoint problem:

$$\varphi_1(y) = (1-y)\,\mathrm{e}^{y}\xi, \tag{33a}$$

$$\varphi_2(y) = m(1+y)\,\mathrm{e}^{-y}\xi, \tag{33b}$$

$$\zeta = -2(1+m)\xi. \tag{33c}$$

Note that both the direct and adjoint eigenfunctions involve a remaining free integration constant due to the homogeneous character of the problem. Both problems give the same eigenvalue:

$$c^{(0)} = 0. \tag{34}$$

Thus at dominant order the wave velocity $(\gamma_1/k)c^{(0)}$ is zero, just as for the clean-interface problem dealt with in [1].

The next step consists in imposing the order-1 compatibility condition:

$$\int_{-\infty}^{0} \mathrm{i}\varphi_1\big(y-c^{(0)}\big)(\psi_1''-\psi_1)\,\mathrm{d}y + \int_{0}^{\infty}\frac{\mathrm{i}r}{m}\varphi_2\left(\frac{y}{m}-c^{(0)}\right)(\psi_2''-\psi_2)\,\mathrm{d}y + \zeta c^{(1)}\eta^{(0)} + \xi\frac{\mathrm{i}\eta^{(0)}}{We_{\mathrm{sw}}} = 0, \tag{35}$$

which immediately gives the order-1 correction of the eigenvalue:

$$c^{(1)} = \frac{\mathrm{i}}{2(m+1)}\left(1+\frac{r}{m^2}-\frac{1}{We_{\mathrm{sw}}}\right). \tag{36}$$

Thus, for zero surface tension, the dimensional growth rate $\gamma_1 Re_{\mathrm{sw}} c_i^{(1)}$ is positive for any density and viscosity ratios, and decreases as $1/k^2$, which means that inertia is destabilizing. In particular, instability persists even for equal viscosities, $m = 1$. Surface tension is stabilising as expected, and its contribution to the growth rate scales as $Re_{\mathrm{sw}}/We_{\mathrm{sw}} \sim k$. These results are similar to the clean-interface case [1,3], except that for the latter case, the growth rate vanishes when viscosities are equal.

### 5.3. *Mechanism of the short-wave instability*

As for long waves, we now explore the mechanism of the short-wave instability, and compare this mechanism to that of the clean-interface flow given in [3]. This mechanism can be described with the help of Fig. 5, which displays typical eigenfunctions for the longitudinal velocities $\psi_{1,2}'(y)$. (The inertial corrections of the eigenfunctions are given in Appendix A.)

The deformation of the interface creates in each fluid a velocity disturbance $u_{1,2}^{(0)} \sim \gamma_{1,2}\eta^{(0)}$ as shown in Fig. 5(a), to which there correspond vorticity disturbances $\omega_{1,2}^{(0)} \sim -ku_{1,2}^{(0)}$. This dominant Stokes flow is in phase with the deformation of the interface.

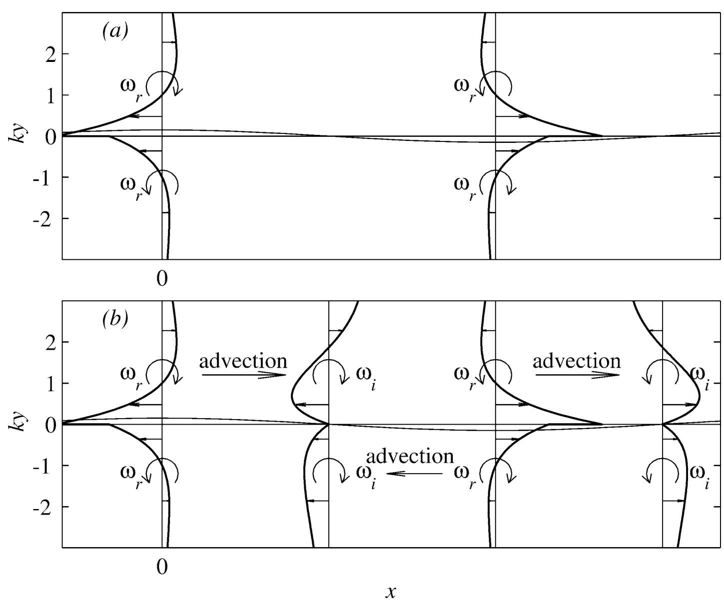


Fig. 5. Eigenfunctions $\psi_{1,2}'(y)$. (a) Dominant Stokes flow $\psi_{1,2}^{(0)\prime}(y)$; (b) Inertial correction $\psi_{1,2}^{(1)\prime}(y)$. Parameters: $m = 0.5$, $r = 1$, $We_{\mathrm{sw}} = \infty$.

In each layer, advection of the vorticity disturbance by the base flow creates a small out-of-phase vorticity component $\omega_{1,2}^{(1)} \sim Re_{1,2}\omega_{1,2}^{(1)}$, where $Re_{1,2} = \rho_{1,2}\gamma_{1,2}/\mu_{1,2}k^2$. For nearly equal viscosities, this vorticity creates a velocity field $u_{1,2}^{(1)} \sim \omega_{1,2}^{(1)}/k$ as shown in Fig. 5(b), and transverse velocities $v_{1,2}^{(1)} \sim \omega_{1,2}^{(1)}/k$, leading to the growth rate:

$$\sigma = \frac{v_1^{(1)}}{\eta^{(0)}} \sim \gamma_1 \frac{\rho_1\gamma_1}{\mu_1 k^2}. \tag{37}$$

When viscosities are quite different, say $\mu_1 \gg \mu_2$, the more viscous fluid slows down the growth of the disturbance. In this more viscous fluid, the pressure gradient $kp_1^{(1)} = kp_2^{(1)} \sim \mu_2 k^2 u_2^{(1)}$ induces velocity components $u_1^{(1)} \sim v_1^{(1)} \sim p_{1,2}^{(1)}/k\mu_1$. This leads to the growth rate:

$$\sigma = \frac{v_1^{(1)}}{\eta^{(0)}} \sim \gamma_1 \frac{\rho_1\gamma_1}{\mu_1 k^2}\frac{r}{m^2}. \tag{38}$$

Both estimates (37) and (38) agree with (36). Thus, the short-wave instability mechanism of a nearly inextensible thin layer is similar to that of a clean interface. However, the zero-longitudinal-velocity condition at the interface induces vorticity disturbances of opposite sign on either side of the interface. Thus, the inertial corrections of vorticity have the same sign on either side of the interface (because advection acts in opposite directions), and the induced transverse velocities are both destabilizing. This explains why instability persists for equal viscosity. For the clean-interface flow, instead, continuity of the longitudinal velocity induces vorticity disturbances with the same sign on either side of the interface, but advection induces vorticities of opposite sign, so that the growth rate vanishes for equal viscosity.

## 6. Summary

The aim of this paper has been twofold: (I) to investigate the stability of a nearly inextensible thin layer in a shear flow, and (II) to illustrate how an adjoint-based approach can be used in order to find the analytical correction of an eigenvalue in a hydrodynamic stability problem.

From the methodological viewpoint, an adjoint-based approach to eigenvalue perturbation expansion allows the eigenvalue perturbation to be obtained at the expense of solving a homogeneous linear system (the adjoint problem) rather than an inhomogeneous one (the first-order direct problem). In addition, different perturbations (inertia and surface tension in this paper, but also others that may become of interest in the future) are easily accounted for by one and the same adjoint solution, without the need of calculating a new particular solution for each case. Thus adjoints, already put to good use in past numerical stability computations [6], can also give a substantial advantage when an analytical solution is sought.

From the physical viewpoint, the conditions under which a three-layer problem can be reduced to a two-layer problem have been preliminarily identified. For a fluid thin layer, these conditions imply small thickness $h_0$ (such that $kh_0 \ll (kh_{1,2})^2$ for long waves and $kh_0 \ll 1$ for short waves), and high viscosity (such that $(kh_{1,2})(kh_0)\mu_0/\mu_{1,2} \gg 1$ for long waves and $(kh_0)\mu_0/\mu_{1,2} \gg 1$ for short waves). For an elastic sheet, equivalent conditions involve the Young modulus.

After reducing the three-layer problem to an equivalent two-layer problem with suitable interface conditions, we have shown that an inextensible thin layer in a shear flow is generally unstable, to both long and short waves. For long waves, the zero-velocity condition at the deformed interface implies that the growth rate is controlled by a pressure-driven flow and not by a shear flow as for a clean interface. For short waves, the zero-velocity condition induces a vorticity disturbance of opposite sign in either fluid. Then, the inertially induced vorticity components are both destabilizing. For both long and short waves, instability persists even for equal viscosity, in contrast with the clean-interface two-layer flow.

## Appendix A. Inertial corrections of the eigenfunctions

For long waves, where the expansion parameter is $\alpha$, the inertial corrections of the eigenfunctions (normalized so that $\eta^{(0)} = 1$ and $\eta^{(1)} = 0$) are given by:

$$\psi_1^{(1)} = \mathrm{i}Re_{\mathrm{lw}}(y+1)^2\left(A_{11}y^4 + B_{11}y^3 + C_{11}y^2 + D_{11}y + E_{11}\right), \tag{A.1a}$$

$$\psi_2^{(1)} = \mathrm{i}Re_{\mathrm{lw}}(y-d)^2\left(A_{21}y^4 + B_{21}y^3 + C_{21}y^2 + D_{21}y + E_{21}\right), \tag{A.1b}$$

$$p_1^{(1)} = p_2^{(1)} = -\frac{1}{10m^2(d^3+m)}\Big(-12{c^{(0)}}^2m^3 - 12c^{(0)}m^3 - 2m^3 + r\left(-7d^3c^{(0)}m + 10d^2{c^{(0)}}^2m^2 - d^5c^{(0)}m - 2d^5{c^{(0)}}^2m + 2d^6 + 4d^6c^{(0)}\right)\Big), \tag{A.1c}$$

with

$$A_{11} = -\frac{1}{60}\left(2c^{(0)} + 1\right), \tag{A.2a}$$

$$B_{11} = \frac{1}{30}c^{(0)}\left(3c^{(0)} + 2\right), \tag{A.2b}$$

$$C_{11} = \frac{1}{60}\left(c^{(0)} + 1\right)\left(3c^{(0)} + 1\right), \tag{A.2c}$$

$$D_{11} = -\frac{1}{30}\left(-5p_1^{(1)} + 6{c^{(0)}}^2 + 6c^{(0)} + 1\right), \tag{A.2d}$$

$$E_{11} = -\frac{D_{11}}{2}; \tag{A.2e}$$

$$A_{21} = -\frac{r}{60m^3}\left(1 + 2c^{(0)}\right), \tag{A.3a}$$

$$B_{21} = \frac{rc^{(0)}}{60d^2m^2}\left(3d^2 + 1 + 6c^{(0)}d^2\right), \tag{A.3b}$$

$$C_{21} = -\frac{r}{60m^3d^2}\left(4d^3c^{(0)}m + 5{c^{(0)}}^2m^2 + 8d^3{c^{(0)}}^2m - 2dmc^{(0)} - d^4 - 2d^4c^{(0)}\right), \tag{A.3c}$$

$$D_{21} = -\frac{D_{11}}{d^3}, \tag{A.3d}$$

$$E_{21} = \frac{E_{11}}{d^2}. \tag{A.3e}$$

For short waves, where the expansion parameter is $Re_{\mathrm{sw}}$, the inertial corrections of the eigenfunctions (normalized so that $\eta^{(0)} = 1$ and $\eta^{(1)} = 0$) are given by:

$$\psi_1^{(1)}(y) = \left(c^{(1)}(1 - y) + \frac{\mathrm{i}}{4}y^2 - \frac{\mathrm{i}}{12}y^3\right), \tag{A.4a}$$

$$\psi_2^{(1)}(y) = \left(c^{(1)}(1 + y) + \frac{\mathrm{i}ry^2}{4m^3} + \frac{\mathrm{i}ry^3}{12m^3}\right), \tag{A.4b}$$

where $c^{(1)}$ is given by (36).

## References

[1] A.P. Hooper, W.G.C. Boyd, Shear-flow instability at the interface between two viscous fluids, J. Fluid Mech. 128 (1983) 507–528.
[2] C.S. Yih, Instability due to viscous stratification, J. Fluid Mech. 27 (1967) 337–352.
[3] E.J. Hinch, A note on the mechanism of the instability at the interface between two shearing fluids, J. Fluid Mech. 144 (1984) 463–465.
[4] F. Charru, E.J. Hinch, 'Phase diagram' of interfacial instabilites in a two-layer Couette flow and mechanism for the long-wave instability, J. Fluid Mech. 414 (2000) 195–223.
[5] E.J. Hinch, Perturbation Methods, Cambridge University Press, 1991.
[6] P. Luchini, A. Bottaro, Gortler vortices: a backward-in-time approach to the receptivity problem, J. Fluid Mech. 363 (1998) 1–23.